Investigative Activity for Discovering Hidden Geometric Properties

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Abstract

We present 5 conservation properties in the context of circles inscribed in a triangle and of circles circumscribing polygons, which are not widely known. The properties were investigated using computerized geometric software. Each property is substantiated by a mathematical proof.

1. Introduction

Even though hundreds of years have passed since the ancient geometers began to discover special geometric properties, many generations of mathematicians have been following them on the same path. Even today [5,13,15] it is possible to discover new things in the rich field of geometry, that challenge the professionals in this field and highlight the beauty contained in the field of geometry, which is a major branch in mathematics.

In fact, each property characterizes conservation through a process of change, and actually represents a theorem in geometry. For such a situation the terms "invariants" and "geometrical invariants" of a figure have been used to refer to certain properties that are retained when some transformations are performed on a geometrical setup [1,7,17]. Geometric properties are perceived in a dynamic geometry environment as invariants through the variation of the figure, "A geometric property is an invariant satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions" [11].

2. Technology in the classroom: The Dynamic Geometry Environment

As is the custom in the modern world which is full of technological tools, today it is impossible to ignore the rapid development of technology and the way it affects almost every facet of life. The education system is no exception, and one certainly cannot disregard the value of technology in teaching mathematics.

Empirical/inductive arguments are the traditional methods are usually used as the first steps for deriving proofs of geometrical problems. However, the advent of dynamic geometry environment (DGE) software serves as an intermediary tool that bridges the gap between a mathematical problem or concept and the associated symbolic proof by providing a clear, visual representation of the relations involved. It is well-established today that DGE has opened new frontiers by linking informal argumentation with formal proof [8,10]. Several researchers conducted extensive exploration into students' behavior when using DGE software and its efficacy in connecting the processes of producing conjectures and proving theorems or statements [2,6].

Introducing DGE software (such as GeoGebra) into classrooms creates a challenge to the praxis of theorem acquisition and deductive proof in the study and teaching of Euclidean geometry. Students/learners can experiment using different dragging modalities on geometrical objects that they construct, and consequently they can infer properties, generalities, and conjectures about the geometrical artifacts. The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and, hence, the students can become convinced that their conjecture will always be true[4].

3. The investigative activity

A part of the investigative activity was carried out by pre-service teachers of mathematics during an advanced course of plane geometry. As part of the course questions were raised, such as: "what happens if one item of data is taken away or added?", "What happens if …… is changed?", "Is the converse theorem also true?", etc [3,9,12].

Fascinating examples are presented on the properties of circles inscribed in a triangle and on circles circumscribing polygons, properties in which we believe are either not known or have not received the proper attention. Links to GeoGebra applets are given in some of the examples, which allow the reader to have a dynamic experience of investigating the property.

The geometric properties presented below were discovered by us by independent investigative activity, and were subsequently adapted for guided investigation by pre-service teachers of mathematics. The main goal was to direct the students towards the discovery of an unknown geometric property using DGE software, the formulation of a hypothesis and its attempted proof.
The students received instructions and hints while the computerized technological tool helped them to determine whether their hypotheses were correct. The applets were prepared beforehand for the students to use during the course.
It was explained to them that the image of the technological tool should not be seen as proof and that formal proof, as is customary in mathematics, is required. It should be emphasized that a large part of proofs obtained by students.
The activity was also used to enrich additional knowledge such as the presentation and use of the Menelaus Theorem, which does not appear in the study program.
All this activity, gave the students an important feeling that they share in research, aimed to the discovery of a "new" geometric properties.

4.1.Theorem 1

In the quadrilateral ABCD it is given that:

The inscribed circle in the triangle ABD and the inscribed circle in the triangle BDC are tangent to each other at the point M on the diagonal BD (see Figure 1).

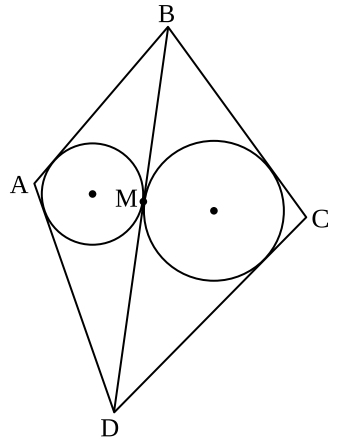

Figure 1 - Two circles tangent to each other at a point on the diagonal of a quadrilateral, and tangent to its sides.

Prove that a circle can be inscribed in the tangential quadrilateral ABCD.

Proof

The proof is based on the theorem:

The lengths of the tangents to the circle that issue from the same point are equal (see Figure 2).

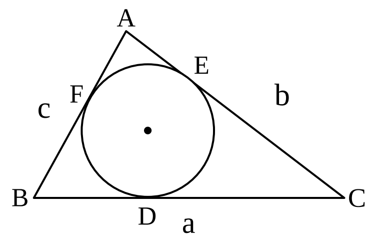

Figure 2 - Tangents to a circle from a single point have equal lengths.

Based on this property, the following relations can be written down:

$$BD = \frac{c+a-b}{2}$$

$$DC = \frac{a+b-c}{2}$$

$$AE = \frac{b+c-a}{2}$$

From these relations we obtain:

In the triangle ABD: $DM = \frac{BD+AD-AB}{2}$

In the triangle BDC: $DM = \frac{BD+DC-BC}{2}$

By equating the two expressions for DM we obtain: $AD+BC = AB+DC$,
which is the condition for the quadrilateral ABCD to circumscribe a circle.

The converse theorem is also true, and its proof is immediate.

An applet was prepared which illustrates the property. The applet can be reached by the following link.

Applet 1: A circle can always be inscribed into this specific quadrilateral
Link 1: https://www.geogebra.org/m/WFXgvvN4

In the applet one can drag each center belonging to the circles that are tangent to each other at the point M, thus changing the magnitudes of their radii. One can also drag the vertices B and D. By dragging, one changes the lengths of the sides of the quadrilateral ABCD while preserving the property that a circle is inscribed in the quadrilateral.

4.2.Theorem 2: A special property of circles inscribed in a triangle
Given is a triangle ABC. We choose a point D on the base of the triangle and draw a straight line AD.

Let (O_1, r_1), (O_2, r_2) and (O, r) be the centers and the radii of the circles inscribed in the triangles ABD, ADC and ABC, respectively, as shown in figure 3.

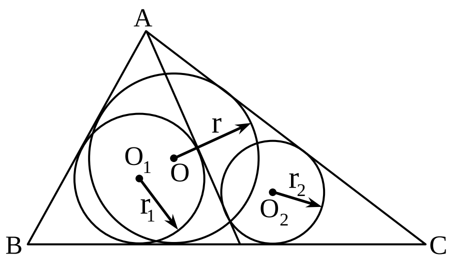

Figure 3 - Three circles inscribed in different triangles.

Prove that $r_1 + r_2 > r$.

We denote the half of the perimeters of the triangles ABD, ADC and ABC by P_1, P_2 and P, respectively, and the areas of the triangles by S_1, S_2 and S, respectively.

Proof
From the calculation of the perimeters and the inequality involving the side lengths of a triangle we have $P_1, P_2 < P$.

From a well-known formula that binds together the area of the triangle, half the perimeter and the radius of the inscribed circle, $r = \frac{S}{P}$, one can write down the following relations:

$$\left.\begin{aligned} r_1 = \frac{S_1}{P_1} > \frac{S_1}{P} \\ r_2 = \frac{S_2}{P_2} > \frac{S_2}{P} \end{aligned}\right\} \Rightarrow \quad r_1 + r_2 > \frac{S_1 + S_2}{P} = \frac{S}{P} = r$$

Applet 2: Presenting the relation between the radii of the circles inscribed in the triangles
Link 2: https://www.geogebra.org/m/tdasPakJ

The applet allows us to drag the point D and each of the vertices of the triangle ABC, and thus to change the side lengths of the triangles, while for each triangle ABC and for each location of the point D the relation $r_1 + r_2 > r$ is conserved.

Follow-up task
We chose any point M inside the triangle. We connect this point with the vertices of the triangles, forming three internal triangles, as shown in Figure 4.

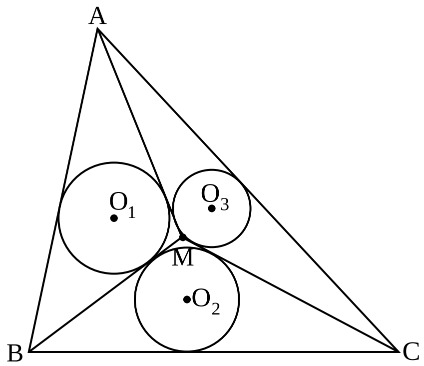

Figure 4 - Three circles inscribed in different triangles.

We denote by r_1, r_2 and r_3 the radii of the circles inscribed in the three internal triangles. Here r is the radius of the circle inscribed in the triangle ABC.

In the same manner we prove that $r_1 + r_2 + r_3 > r$.

4.3.Theorem 3

a. Each polygon with an **odd** number of sides that is inscribed in a circle and has equal angles is a regular polygon.

   <u>Proof</u>
   Given: $\angle A_1 = \angle A_2 = \angle A_3 = ... = \angle A_{2n+1}$.
   The quadrilateral $A_1A_2A_3A_4$ is circumscribed by a circle.
   $\angle A_3A_4A_1 = 180° - \angle A_2$
   Since $\angle A_2 = \angle A_3$, it follows that
   $\angle A_3 + \angle A_3A_4A_1 = 180°$ (see Figure 5).

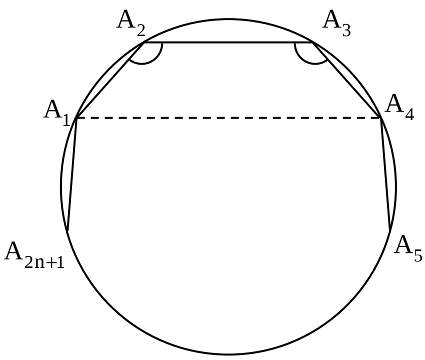

Figure 5- Polygon with an odd number of sides, inscribed in a circle.

Therefore the quadrilateral $A_1A_2A_3A_4$ is an isosceles trapezoid, in other words: $A_1A_2 = A_3A_4$.

In the same manner we continue and obtain:
$A_1A_2 = A_3A_4 = A_5A_6 = \ldots = A_{2n+1}A_1 = A_2A_3 = A_4A_5 = \ldots = A_{2n}A_{2n+1}$

Therefore all the sides are of equal length, and the polygon is regular.

To illustrate the conclusion obtained from the proof, we take a pentagon that is circumscribed by a circle and all of whose angles are equal, as shown in Figure 6.

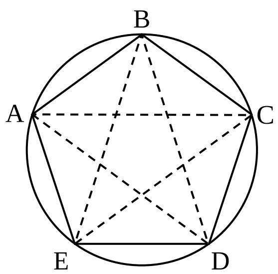

Figure 6 - Regular pentagon inscribed in a circle

As proven, when the adjacent angles are equal, isosceles trapezoids are formed, and therefore: $AB = CD = AE = BC = ED$.

b. In a polygon with an even number of sides, it is possible for it to have equal angles without being regular (for every even n).
The clearest example is the infinite number of rectangles which can be circumscribed by a circle: all the angles equal 90° but the side lengths are not equal.

We shall show that it is possible to construct a polygon with 2n sides (for any n), which is inscribed in a circle and all of whose angles are equal.

The example for such a construction will be carried out for a polygon with six sides. We first construct an equilateral triangle ABC. On the arc $\overset{\frown}{AB}$ we choose a point E, from which we construct the equilateral triangle EFG, as shown in Figure 7.

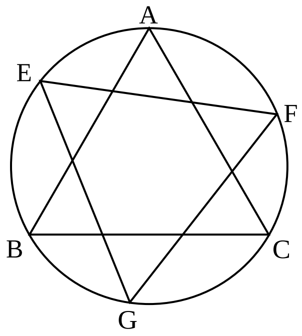

Figure 7 - Polygon with an even number of sides, which is not regular and is inscribed in a circle.

From the construction we obtain:
$\overset{\frown}{EA} = \overset{\frown}{FC} = \overset{\frown}{GB}$, $\overset{\frown}{AF} = \overset{\frown}{CG} = \overset{\frown}{BC}$.

From the theorem that on equal arcs rest equal inscribed angles, we obtain that the angles of the hexagon ABCGBEA are equal, but it is not necessarily a regular hexagon.
In general, we construct a regular polygon of n sides inscribed in a circle, with its vertices A_1 , A_2 , A_3 , ... , A_n . Starting from some point B_1 on the arc $\overset{\frown}{A_1A_2}$ (not in the middle of the arc) we construct another regular polygon whose vertices are B_1 , B_2 , B_3 , ... , B_n .

The polygon whose vertices are $A_1B_1A_2B_2A_3B_3...A_nB_n$ is not necessarily regular but all its angles are equal.

Applet 3: Presentation of an octagon with a circumscribed circle, that is not necessarily regular but all its angles are equal.
[Link 3: https://www.geogebra.org/m/h6xusKsr](https://www.geogebra.org/m/h6xusKsr)

The applet shows an octagon with the circumscribed circle, which is not necessarily regular and each of whose angles is 135°. Using the toolbar, one can change the lengths of its sides. Half of the sides have one length and the other half of the sides have a different length, where the sides of different length alternating.

Applet 4: Presentation of the method of constructing an octagon with a circumscribed circle, that is not necessarily regular but all of whose angles are equal.
[Link 4: https://www.geogebra.org/m/qS99awyx](https://www.geogebra.org/m/qS99awyx)

The applet describes how one should build an octagon with the circumscribed circle, which is not necessarily regular but has equal angles. We construct two inscribed squares in the circle – one fixed and the other rotated using the toolbar about the center of the square. The vertices of the fixed square in the vertices of the rotated square are the vertices of the octagon.

4.4.Theorem 4

Two chords are given in a circle with a radius R: AC and BD, which are perpendicular to each other. Prove that $AB^2 + CD^2 = 4R^2$ (see Figure 8).

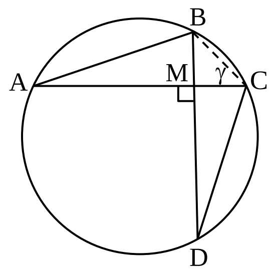

Figure 8 - Relations between the lengths of perpendicular chords in a circle.

Proof

We denote: $\angle BCM = \gamma$, therefore $\angle CBM = 90° - \gamma$.

From the Law of Sines in the triangle ABC we have: $\frac{AB}{\sin\gamma} = 2R \quad \Rightarrow \quad AB = 2R\sin\gamma$

From the Law of Sines in the triangle BCD:

$\frac{CD}{\sin(90° - \gamma)} = 2R \quad \Rightarrow \quad CD = 2R\sin(90° - \gamma) = 2R\cos\gamma$

Hence: $AB^2 + CD^2 = 4R^2(\sin^2\gamma + \cos^2\gamma) = 4R^2$

An elegant proof in the style of "a Proof without Words" was published in [17].

Theorem 5: **Special property**

In some triangle ABC, AD, AE and AF are the altitude, the angle bisector and the median (respectively) that issue from the vertex A.

Let the point O be the center of the circle inscribed in the triangle.

The continuation of the straight line FO intersects the attitude AD at the point G (see Figure 9).

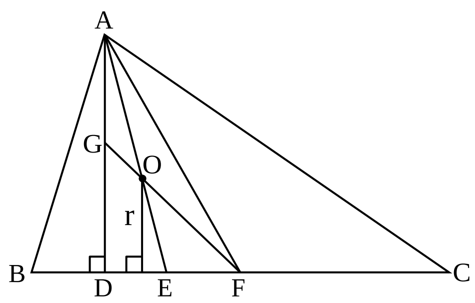

Figure 9 - Segment in a triangle whose length is equal to the radius of the incircle.

Prove that AG = r, where r is the radius of the circle inscribed in the triangle.

Proof

We denote the sides of the triangle as follows: $AB = c$, $AC = b$, $BC = a$.

Claim 1: $\frac{DF}{EF} = \left(\frac{b+c}{a} \right)^2$.

Proof of the claim

From the angle bisector Theorem, we have $\frac{c}{b} = \frac{BE}{EC}$, and by mathematical manipulation we obtain:

$$BE = \frac{ca}{b+c} \quad (1)$$

$EF = BF - BE = \frac{a}{2} - \frac{ca}{b+c}$ and hence:

$$EF = \frac{a(b-c)}{2(b+c)} \quad (2)$$

From an extension of the Pythagorean Theorem, we get $BD = \frac{c^2 + a^2 - b^2}{2a}$, and therefore

$$DF = \frac{a}{2} - BD = \frac{b^2 - c^2}{2a}$$

$$DF = \frac{b^2 - c^2}{2a} \quad (3)$$

From the relations (2) and (3) it follows that $\frac{DF}{EF} = \left(\frac{b+c}{a} \right)^2$.

By considering the triangle ADE and the straight line FG that intersects it, from Menelaus' Theorem there holds:

$$\frac{DF}{EF} \cdot \frac{EO}{OA} \cdot \frac{AG}{GD} = 1 \quad (4)$$

BO is an angle bisector in the triangle ABC, and therefore from the angle bisector Theorem and (1) it satisfies: $\frac{BE}{AB} = \frac{OE}{AO} = \frac{\frac{ca}{b+c}}{c} = \frac{a}{b+c}$

We substitute the relation $\frac{DF}{EF}$ (Claim 1) in the ratio $\frac{OE}{AO}$ in (4): $\left(\frac{b+c}{a} \right)^2 \cdot \frac{a}{b+c} \cdot \frac{AG}{GD} = 1$

Therefore: $$\frac{AG}{GD} = \frac{a}{b+c} \quad (5)$$

from which we have: $$\frac{AG}{GD} = \frac{a}{b+c+a} \quad (6)$$

but: $\frac{OE}{AO} = \frac{a}{b+c}$ and hence

$$\frac{OE}{AE} = \frac{a}{b+c+a} \quad (7)$$

From Thales' Theorem in the triangle ADE we obtain: $\frac{OE}{AE} = \frac{r}{AD} = \frac{a}{b+c+a}$

But from relation (6) we have that $\frac{AG}{AD} = \frac{r}{AD}$, and therefore AG = r.

Note:

For the case where $\angle B > 90°$, the altitude AD goes outside the triangle, as shown in Figure 10. In that case, $BD = \frac{b^2 - a^2 - c^2}{2a}$, and then $DF = BD + \frac{a}{2} = \frac{b^2 - c^2}{2a}$, as in the case where $\angle B < 90°$.

Therefore the property holds in general case for any triangle: acute, right or obtuse.

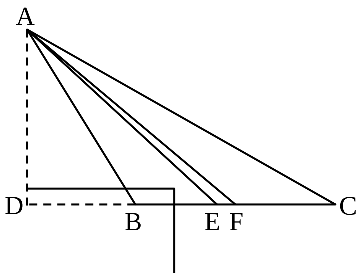

Figure 10 - Example of a segment in an acute-angled triangle whose length is equal to the radius of the incircle

Methodical note:
The proof of this property was an opportunity to show the students the important Menelaus' Theorem, and for illustrating its applications. As far as we know, this theorem is not included in the program of studies of high schools or in the program of studies of academic institutions in the routes for training pre-service teachers of mathematics. Mathematical enrichment and expansion of the mathematical toolbox constitute the added value of teacher training.

Applet 5: The segment marked off on the attitude always equals the radius of the inscribed circle in the triangle
[Link 5: https://www.geogebra.org/m/qkhbMEy7](https://www.geogebra.org/m/qkhbMEy7)

To illustrate the property an applet was prepared in which the vertices of the triangle can be dragged thereby changing the lengths of its sides and the magnitudes of its angles. For each triangle the values of r and AG appear on screen, and their values are indeed equal.

Follow-up tasks of this task
The discovery of the property that AG = r prompted the raising of additional questions which led to additional properties. It is clear that this property is true for any segment marked off in the same manner on the altitudes to the sides b and c.

From each vertex of the triangle used as the center we draw circles with the radius r, as shown in figure 11.

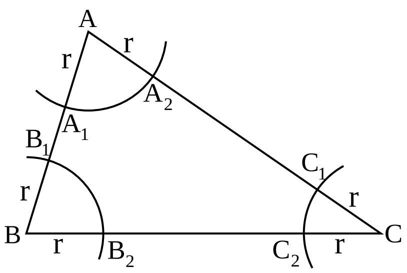

Figure 11 - Segments cut off by the radius of the incircle on the sides of a triangle, when the former is drawn from the vertices.

These circles mark segments off the sides of the triangle. The relations between the lengths of the segments express the nature of the original triangle ABC.

Claim A
The circles with the radius r whose centers are of the vertices A, B and C do not intersect, or, formulated otherwise: in each triangle the segments A_2B_1 , B_2C_1 , C_2A_1 form on the sides.

Proof of the claim:
Each of the vertices satisfies h_a , h_b , $h_c > 2r$ [14].

Each of the sides satisfies the relation: $a > h_a$, $b > h_b$, $c > h_c$, and therefore a , b , $c > r$.

Claim B
B.1. If the segments A_2B_1 , B_2C_1 , C_2A_1 satisfy the triangle inequality, then the triangle ABC is acute angled.

B.2. If the segments satisfy the following relation: $A_2B_1 = B_2C_1 + C_2A_1$, then the triangle is right angled ($\angle C = 90°$).

B.3. If the segments A_2B_1 , B_2C_1 , C_2A_1 do not satisfy the triangle inequality, then the triangle ABC is obtuse angled.

Proof
For the proof we make use of Figure 12, which shows the triangle and the circle inscribed in it.

For the case B.1, where $A_1C_2 + B_2C_1 > A_2B_1$, one can write down: $b - 2r + a - 2r > c - 2r$, and hence: $a + b - c > 2r \quad \Rightarrow \quad p - c > r$.

The segment p – c is the segment DC, shown in Figure 12.

$$\mathrm{tg}\frac{\gamma}{2} = \frac{r}{p-c} \quad \Rightarrow \quad \mathrm{tg}\frac{\gamma}{2} < 1 \quad \Rightarrow \quad \gamma < 90°$$

In other words, $\angle C < 90°$. In the same manner we prove that the angles $\angle A$ and $\angle B$ are smaller than 90°.

For the case B.2, where $B_2C_1 + C_2A_1 = A_2B_1$, we obtain p – c = r,

and therefore $\mathrm{tg}\frac{\gamma}{2} = 1 \quad \Rightarrow \quad \angle C = 90°$

For the case B.3, where $A_1C_2 + B_2C_1 < A_2B_1$, we obtain p – c < r,

and therefore $\mathrm{tg}\frac{\gamma}{2} > 1 \quad \Rightarrow \quad \angle C > 90°$

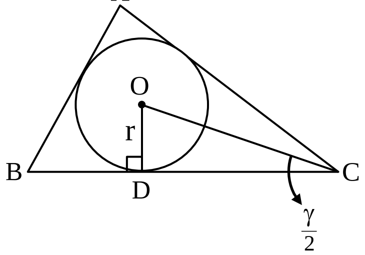

Figure 12 - Marking segments in a circle.

Note:
Case B.2 is an interesting case, since there we have:

$$a+b-c=2r \quad \Rightarrow \quad a+b=c+2r=2R+2r$$

where R is the radius of the circumcircle of the triangle. In other words: in a right-angled triangle the sum of the lengths of the diameters of the incircle and the circumcircle is equal to the sum of the lengths of the legs [16].

Applet 6: Presentation of the transition from an acute-angled triangle to an obtuse-angled triangle through the lengths of the segments marked off on the sides

Link 6: https://www.geogebra.org/m/uFeKeS5m

In the applet one can drag each of the vertices of the triangle the changing the names of its sides. During each stage the following values appear on the screen: the lengths of the segments A_2B_1 , B_2C_1 , C_2A_1, the sum of the lengths of the segments $B_2C_1 + C_2A_1$, and the magnitude of the angle $\angle ACB$. When the vertices are dragged, and we pass from an acute-angled triangle to an obtuse-angled triangle, we can see the relation between the lengths of the segments A_2B_1 , B_2C_1 , C_2A_1. When $\angle ACB=90°$, there holds $A_2B_1 = B_2C_1 + C_2A_1$.

Note:
When the triangle passes from an acute angled triangle to an obtuse angled triangle, the property in Theorem 5 remains valid. In other words, the straight line that issues from the middle of a side and passes through the center of the inscribed circle marks off the segment equal in length to the radius of the circumscribing circle on the altitude to the side.

4.5.Conclusions:

1. Theorems were presented, which originally were discovered with the aid of DGE software, and which were used for conducting investigative activities by pre-service teachers of mathematics.
2. The large majority of the students who took part in the activity was very interested and was very impressed by the use of the applets for the discovery of hypotheses.
3. Difficulties were found in the students' ability to prove mathematically the correctness of the hypotheses after the visual affirmation of their correctness was obtained using the DGE software. These difficulties were overcome through providing hints, refreshing and enriching the knowledge. This allowed the required formal proof to be attained, as mathematics dictates.
4. The large majority of the students were willing and eager to carry out guided investigation with the aid of the computerized technological tool in their specialization classes with the purpose of increasing the motivation and enjoyment of students from studying mathematics, and for highlighting its beauty.
5. An important conclusion is that before investigative activities of the same style one must prepare well the investigated configuration in order to focus the activity and preventing dead ends.

5. Summary

Even though the field of Euclidean geometry "Has been ploughed thoroughly" over many generations during which mathematics has developed, even today one can find in it surprising properties that can serve as an additional layer in the extensive and fascinating jigsaw puzzle constructed from axioms and theorems. The computerized tool acts as a “search engine” assisting in the discovery of these properties.

6. References

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